# COMPARISON OF OPTICAL AND ELASTIC BREWSTER'S ANGLES TO PROVIDE INVUITIVE INSIGHT INTO PROPAGATION OF P- AND S-WAVES

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## ABSTRACT

The Brewster's (polarization) angle of reflection in optics is well understood and leads to distortion, including polarization filtering, of reflected waves. Further, the values of the Brewster's angle (a zero crossing in amplitude) are closely related to the contrast in physical properties across the reflecting interface. I investigate elastic analogues to the optical Brewster's angle for the case of incident seismic shear waves at a reflecting interface. Both optical (light) and elastic shear waves are characterized as transverse waves, and both are subject to polarization distortion upon reflection. For this exploratory study, I limit the seismic case to situations of only two reflected or transmitted waves where one of the waves vanishes: 1.) SH-SH reflection-refraction across a solid-solid interface where the reflected SH wave vanishes and 2.) SV-P reflection (mode conversion) at a free interface where the reflected SV wave vanishes. In the optical case, the rays defining the refracted and reflected waves at the Brewster's angle are at 90° to each other. In the shear-wave examples, the reflected and refracted SH waves are normal to each other only if there is no density contrast. For the free surface, the rays for the incident SV and reflected P waves are at right angles only for a Poisson's solid, $\lambda = \mu$. Understanding these geometric relations should improve our intuitive insight into reflection, refraction and mode-conversion processes for P, SV and SH waves and improve interpretation of contrasts in elastic properties, including anisotropic conditions. Further investigations into characterizing contrasts in elastic properties through reflection/refraction/mode-conversions processes are continuing.

## INTRODUCTION

All light (EM) waves are transversely polarized and thus share some propagation properties with elastic shear waves. In particular, optical waves reflecting from an interface will experience changes in their polarization upon reflection. This effect may be beneficial in addressing propagation constants, or it may be harmful in that it distorts the polarization of the reflected wave. Here we consider the elastic analogy of the Brewster's (or polarization) angle for light waves and address differences and similarities in the case of seismic (elastic) waves. We attempt to identity and isolate wave properties associated with the elastic Brewster's angle $j_B$ for incident shear waves and provide some physical insight into seismic wave properties and the contrasts in elastic properties controlling them.

Brewster's angle (polarization angle) occurs for light reflecting from an interface such as air over a glass plate. This angle may be defined as the angle—for a particular polarization—where energy is entirely transmitted into the refracting medium, and thus has zero reflectivity. The polarization where this occurs, in terms of the nomenclature used in seismology, is the SV polarization—or transverse polarization in a plane normal to the reflecting interface. Figure 1, from a basic optics textbook, illustrates the concept. If the incident energy is arbitrarily polarized with components in both the equivalent SH and SV polarization, there is no reflection of the SV component. Thus, an arbitrarily polarized incident wave will result in a purely SH polarized reflected wave. Further, the physics of optical wave propagation requires that the angle $(\phi + \phi')$ between the transmitted and reflected ray will always be 90°. This 90° angle (The polarization of the two light waves are orthogonal) results in the incidence Brewster's angle being described by: $j_B = \tan^{-1}(\upsilon_1 / \upsilon_2)$ where $\upsilon_1$ and $\upsilon_2$ are the speed of light in the upper and lower media, respectively. This reflection relation has a similarity to the $\upsilon_1$ and $\upsilon_2$ critical refraction angle (for $\upsilon_1 > \upsilon_2$ ) $i_c = \sin^{-1}(\upsilon_1 / \upsilon_2)$, and shows how the Brewster's $j_B$ angle approaches $i_c$ as $\upsilon_1 / \upsilon_2$ increases. If $\upsilon_2 < \upsilon_1$, there is no critical angle.

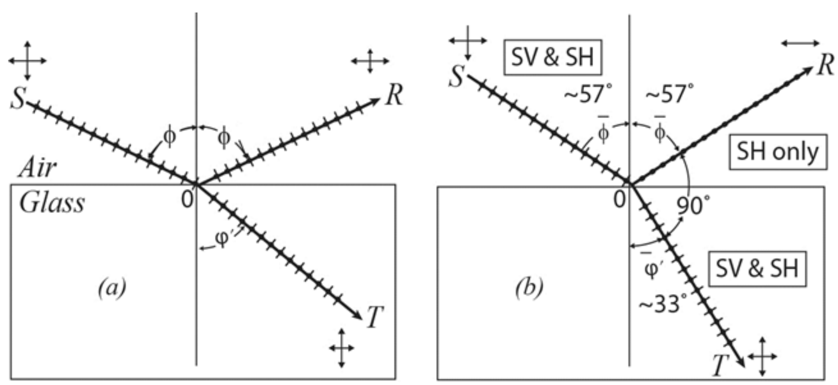


**Figure 1. Optical polarization by reflecting and refraction (From Jenkins and White, 1957).**

## THEORY and METHOD

### SH-SH

Perhaps the geometrically simplest analogy to the optical reflection process is the SH-SH reflectivity. Note that the SH wave does have a zero-crossing at relatively modest angles of incidence. In the SH elastic case, the incident, reflected and transmitted SH waves all maintain the same polarity (normal to the vertical plane) for all propagation angles. For the case where the reflected wave vanishes ($j_B$, the elastic Brewster's angle) we need only consider the transmitted SH wave. Aki and Richards (2002) do provide a relation for the SH-SH reflectivity:

$$R_{SH-SH} = \frac{\rho_1\beta_1 \cos j_1 - \rho_2\beta_2 \cos j_2}{\rho_1\beta_1 \cos j_1 + \rho_2\beta_2 \cos j_2} \tag{1}$$

where $\rho_1$ and $\rho_2$ are the density in the upper and lower layers, $\beta_1$ and $\beta_2$ are the shear-wave velocities in the upper and lower layers and $j_1$ is the angle of incidence (and reflection) and $j_2$ is the angle of refraction. For $R_{SH-SH} = 0$, we have no reflection, and thus $j_1$ will be indicated by $j_B$, the elastic Brewster's angle. At this angle, we can set the numerator in Equation 1 to 0 (because the denominator never vanishes), and we have a relation:

$$\sin^2 j_B = \frac{\left(Z_2^2 / Z_1^2\right) - 1}{\left(Z_2^2\beta_2^2\right) / \left(Z_1^2\beta_1^2\right) - 1} \tag{2}$$

where $Z$ is the shear impedance, $\rho\beta$. This is an equation very similar to the purely acoustic case, and has some similar consequences. Basically, for increases or decreases in both the elastic impedance (which is dependent upon velocity) and the seismic velocity, there is always a real Brewster's angle. A summary of this result is shown in Figure 2, where the horizontal axis is the Brewster's angle and the vertical axis is the contrast in shear velocity, $\beta_2 / \beta_1$. Each trace of the plot is for an individual density contrast, $\rho_2 / \rho_1$. Interestingly, for no density contrast ($\rho_2 / \rho_1 = 1$), the relation between the Brewster's angle matches that of the optical case, $\tan^{-1}(j_B) = \beta_1 / \beta_2$. For this case, just like the optical case, the angle between the refracted ray and the reflected (zero amplitude) ray is 90°. This is true even though the polarization of the particle motion associated with all the rays is universally unchanged. What does change is the direction of the orientation of the spatial derivative defining the strain associated with the shear-wave propagation. Note that in Figure 2, the range in values of Brewster's angle for no density contrast is from about 35° to 55° for the +/–25% range in velocity contrast. For a +/–10% velocity contrast, the relatively narrow range in $j_B$ is only 40°–50°. Density contrasts do expand this range somewhat. Nonetheless, we do see rather restricted ranges in possible values of $j_B$.

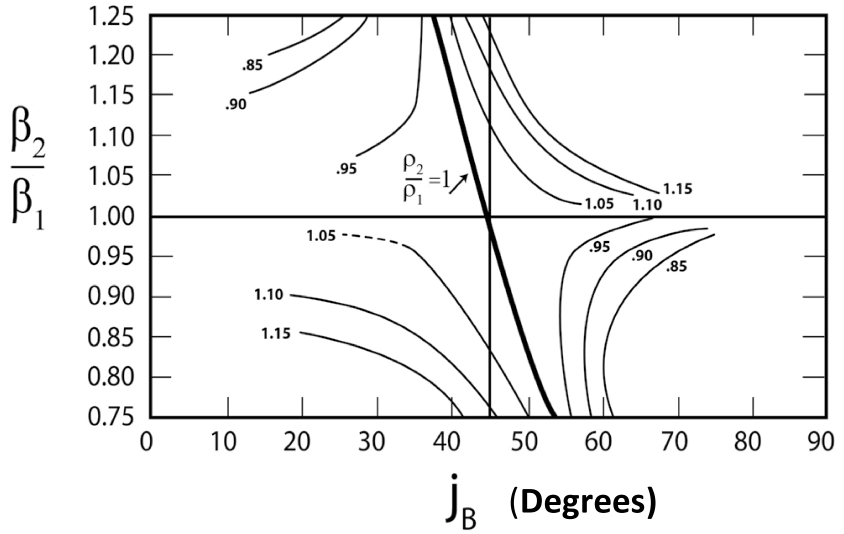


**Figure 2. Brewster's angle vs. velocity contrast for SH-SH waves. The individual lines are for various density contrasts $r_2/r_1$. For no density contrast ($r_2/r_1$ = 1), the angle of reflection is normal to the angle of reflection ($\tan^{-1} j_B = b_1/b_2$) —as is the case for optical waves.**

## SV-SV and SV-P

In the elastic case of a seismic SV wave interacting with a solid/solid boundary, there are four resultant waves, and thus the situation is far more complex. This result occurs because of mode conversion from SV waves upon both reflection and transmission to P-waves. Because the P waves have faster propagation velocities than the SV-waves, there will always be critical angles (up to three) for incident SV waves. The existence of these critical angles may have significant effects on the reflected and transmitted waves. One of the motivations in addressing a physical insight of the Brewster's angle phenomena is to expand our understanding of the observation that the value of angle $j_B$ for an incident SV wave is, for most contrasts in P- and S-impedances, very nearly constant (Krohn, 1988; Lyons, 2006; and Campbell and Tatham, 2011). Further, the sensitivity of this angle to contrasts in physical properties, as small as the variations may be, could prove useful in the interpretation of many rock properties, including anisotropy.

## FREE SURFACE

To address the vanishing SV case, similar to the optical case, we consider in incident SV wave reflecting at a free interface. For the free-air case, there is no transmitted energy, but there are two reflected waves, SV and P. We consider the case where the reflected SV energy is zero, for the incidence angle $j_B$, all the incident SV energy is reflected in the mode-converted P-wave.

This is analogous to the optical case where there are only two possible waves—with one being zero energy. This also relates to the SH-SH case where no reflected shear-wave energy exists.

Further, by considering the internal reflections at a free surface, the dependence on density in the reflection coefficients disappears. In addition, in the optical case, the speed of light in the refracting medium is usually lower than in air, so there are no critical angle effects. In the elastic case, the P-wave velocity a is always greater than β, the shear-wave velocity; thus there is always an internal critical angle associated with the internally refracted P-wave. This adds an additional complexity to the seismic situation. In addition, unlike the solid/solid case, the free surface represents a very strong 'contrast' in elastic properties and density, and may represent a rather extreme case compared to the solid/ solid case. Nonetheless, the simplicity of just two possible waves and boundary conditions, as well as the independence of density, offer an opportunity to focus on some of the physical interactions involved and provide preliminary insight into the fundamental processes.

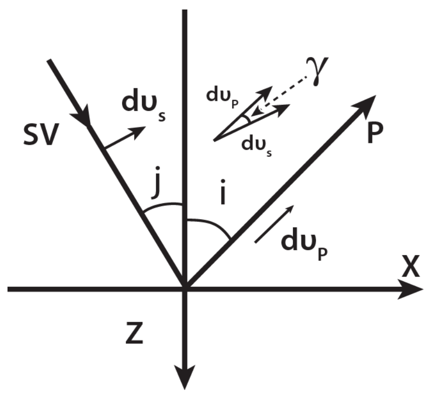


**Figure 3. Reflection of a mode-converted SV-P wave at a free surface. Note particle motion displacements $du_s$ and $du_p$ for SV & P waves and g, the angle between the $du_s$ and $du_p$ displacement vectors.**

Figure 3 illustrates this simple free-surface geometry, including the definition of the angle between the polarizations of the particle displacement of the incident SV and reflected P-waves. For the optical case, the angle between the refracted and reflected rays is 90°, which is the same as $\gamma$= 90° for the polarizations of the light. This is not true for the elastic case. An example of an SV-SV reflection coefficient from a free surface of a Poisson solid ($\alpha/\beta = \sqrt{3}$) is shown in Figure 4. Note that there are two angles (30° and 34°) where the SV reflection vanishes. The larger angle is very close to the critical angle, 35°. Interestingly, for this case of a Poisson solid, $\gamma = 0$. We will consider a full range of values of Poisson' s ratio and examine the interactions between $j_B$.

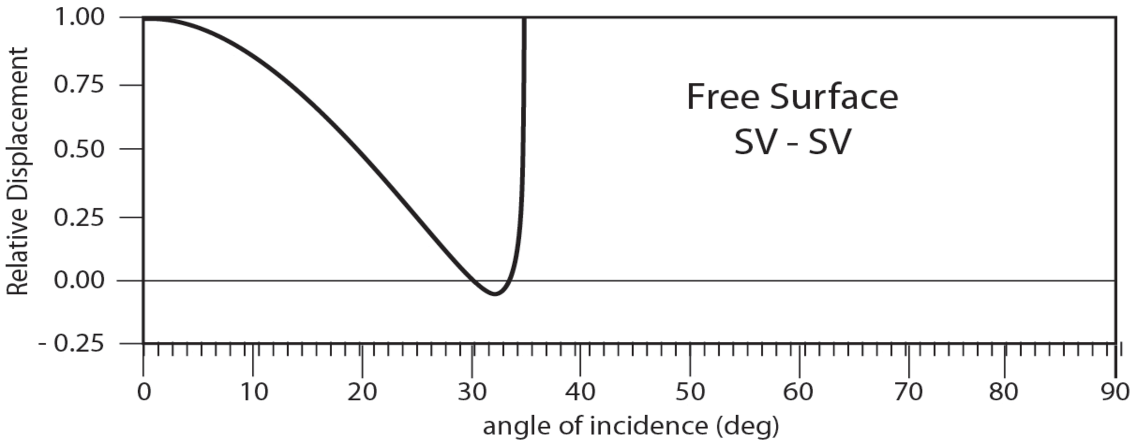


**Figure 4. Reflection amplitudes at free surface for an SV wave propagating in a Poisson's solid. After Cerveny (2001)**

Another difference between the optical and free-surface elastic case is the polarization of the waves themselves. For the elastic case at $j_B$, the energy in the transverse polarization of the incidence SV wave is entirely transferred to the longitudinal polarization associated with the reflected P-wave. As mentioned, the angle between these two (SV & P) polarizations is generally not zero. Aki and Richards (2002) do provide a relation for the SV-SV reflection coefficient:

$$R_{SV-SV} = \frac{\left(\frac{1}{\beta^2} - 2p^2\right)^2 - 4p^2 \frac{\cos i}{\alpha}\frac{\cos j}{\beta}}{\left(\frac{1}{\beta^2} - 2p^2\right)^2 + 4p^2 \frac{\cos i}{\alpha}\frac{\cos j}{\beta}} \tag{3}$$

where i and j are the ray angles for P and S waves and p, the ray parameter, = sin i / a = sin j / b For Rsv-sv = 0, we can set the numerator to zero (because the denominator never vanishes),and solve for $j_B$. The solution is a cubic in $\text{Sin}^2\, j_B$, consistent with the possibility of more than one root. The result of calculating $j_B$ is plotted as a function of a/b in Figure 5. In this case, we find two real values of $j_B$ and only a limited range of values of a/b where the $j_B$ exists: √2 < a/b < ~1.76. Significantly, this upper value is greater than √3, thus probably not related to the simplicity of a Poisson solid where l = μ. The angle $j_B$ at this limit point is about 32°. For the larger values of $j_B$ (> 34°), the second Brewster's angle approaches the critical angle and merges with the critical angle at a/b = √2. Also included in Figure 5 (insert) is the variation in g for the range of 26 to about 42 where there is a zero crossing of the SV-SV reflection. The range in is fairly large, -30° to over + 30°.

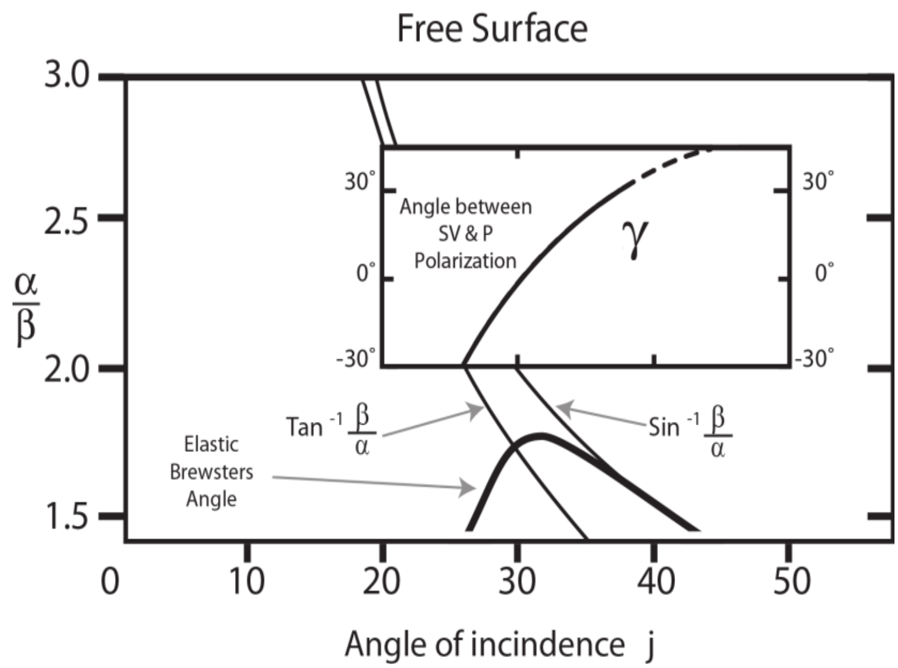


**Figure 5. Angle of incidence j where the reflectivity Rsv-sv of the reflected SV polarized shear wave from a free-surface interface vanishes, as a function of $\alpha/\beta$. Also included is $\gamma$ for the range of j where there is a zero crossing in Rsv-sv (only P-wave reflected).**

Unlike the optical case, there is not a simple relation between $j_B$ and i. We did derive a Brewster-like expression in terms of i and $j_B$:

$$i = \sin^{-1}\left(\frac{1-\cos 2j_B}{\cos 2j_B}\right)\tan 2j_B \tag{4}$$

The relation of I with $\alpha/\beta$ is included in terms of $\alpha/\beta = \sin i / \sin j_B$.

## DISCUSION

Note that for values of $\alpha/\beta >\sim 1.76$, there is no polarity reversal for the reflected SV wave, and hence no associated Brewster's angle.(Figure 5) In the case of a solid/solid interface, there is always a change in the polarity of a reflected shear wave between normal and grazing incidence for an increase in the shear wave impedance, and thus there is always a zero-crossing associated with the wave. For very large velocity contrasts, however, this angle may become complicated with one of three possible critical angles. Further, for decreases in shear impedance less than about 20%, there is nearly always a zero crossing polarity of the SV wave. Thus, for most situations of a solid/solid interface, there is an elastic Brewster's angle associated with incident

SV waves. In this sense, the extreme 'contrasts' of the free surface model and the limited occurrences of Brewster's angle do not fully represent an analogue for the solid/solid model. The simplicity of the small number of waves and no contrast in density, however, does allow us to examine individual angles more thoroughly. One further approach to increasing our insight to reflection processes, in both the free surface and solid/solid cases, is to address the boundary conditions directly. The boundary conditions at the free surface require that the total traction at the surface resulting from the sum of the incident and reflected waves be zero. We say that the angle $j_B$ of the incident SV wave is such that the reflected SV wave has zero amplitude. This means that the total traction on the free surface due to the combination of the incident SV wave and the converted P-wave must be zero. In the same way, when the incident SV wave is at normal ($j=0$) incidence, the combination of the incident and reflected SV waves causes the traction on the free surface to be zero because there is no converted P-wave at normal incidence angle. Here, we have only addressed incident SV waves. In many cases, there are indeed elastic Brewster's angles associated with incident P-waves, but there is not the same 'certainty that they occur as there is with the SV wave. Levin (1986), considering discussions of AVO effects and zero-crossings for Type I and II gas sands, does address the Brewster's angles for P-P reflectivity. He shows two values for $j_B$, where it exists, for several combinations in velocity and density contrasts across a solid/solid interface—not unlike the result we observe in Figure 5 for the free surface case. A brief perusal of the acoustic literature revealed publications of elastic Brewster's angle studies for incident acoustic (P-) waves. They included studies of solid/solid and solid/fluid interfaces and cases of anisotropy and viscoelasticity. Thus, this direction is not without some precedence. The missing element appears to be some intuitive insight into what physical processes occur at the reflecting boundaries. The free surface case, as extreme as it is, offers very simple geometry to attack this issue—but we must not lose sight of the more complex solid/solid interface.

## CONCLUSIONS

To date, we have identified geometrical and physical aspects of the SH-SH reflectivity and SV-P reflection-transmission-mode conversion processes that provide a focus for more detailed investigation into the subject interactions. The SH-SH case is almost a stand-alone condition. Some results to note (in Figure 5) for the free surface case are that for a minimum meaningful value of $\alpha/\beta$ of $\sqrt{2}$, there is a zero crossing (Brewster's angle $j_B$) at 26°. A second zero crossing in SV reflectivity, $j_B = 45°$, is coincident with the critical angle for this particular velocity ratio. The angle $\gamma$ between the polarization of the incident SV wave displacement vector and the reflected P-wave displacement is about -27°. At the second value of $j_B$, coincident with the critical angle, $\gamma = +45°$. As the velocity ratio increases, there are two values of $j_B$, although the second zero crossing is very close to the critical angle. At $\alpha/\beta = \sqrt{3}$ (a Poisson solid where $\lambda=\mu$), the first value of $j_B = 30°$, and $\gamma = 0°$. In this case, the incident SV and reflected P rays are at 90°. The second $j_B$ is at 34°, and $\gamma = 16°$. As $\alpha/\beta$ increases in value above $\sqrt{3}$, associated with a Poisson solid, we reach a limit in $\alpha/\beta$ of 1.76, above which there is no zero crossing in the SV reflection. At this point, $j_B = 32°$ and $\gamma = 11°$. This value of $\alpha/\beta$ is well within the range of common sedimentary rocks. Keep in mind, however, that this is free interface, with its associated limits in representing subsurface reflections. Additional efforts will include developing a deeper insight into the significance of the angle $\gamma$, the nature of the boundary conditions on these effects and extension to full solid/solid interfaces.

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